

# Quadrupole magnet

Kai Hock

Cockcroft Institute / Liverpool University

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I have been fascinated by quadrupole magnets ever since I first learnt about them in accelerator physics. So simple, yet so essential a part of the accelerator. The section on bending magnets was about the dipole, a uniform magnetic field that is used to bend the electrons. The purpose of the quadrupole magnet is to focus the electrons, like a lens focuses light.

There are many ways to make a quadrupole magnetic field. A simple way is shown in fig. 1.

The way it is normally arranged on a damping ring is shown in fig. 2. An electron beam perpendicular to the page would go through the centre. The red colour bits represent the magnet pole faces that are shaped in such a way that the field quality is optimised. But you can see that the field pattern is essentially the same as that created by the four magnets in fig. 1.

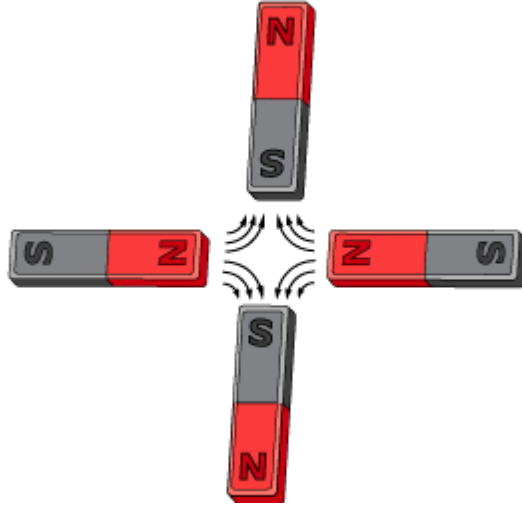


Figure 1: To make a quadrupole field [1].

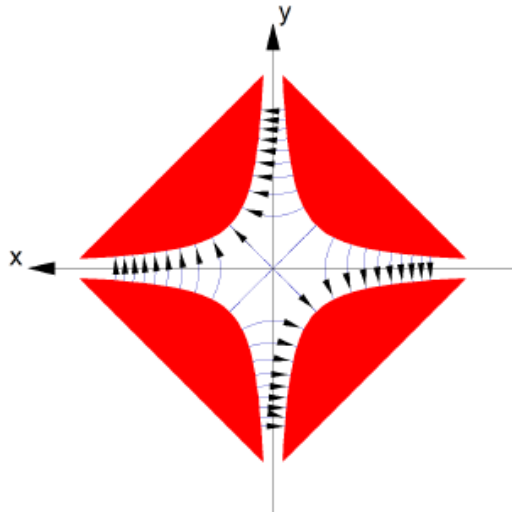


Figure 2: Quadrupole in an accelerator [2]. The  $z$  axis goes into the page.

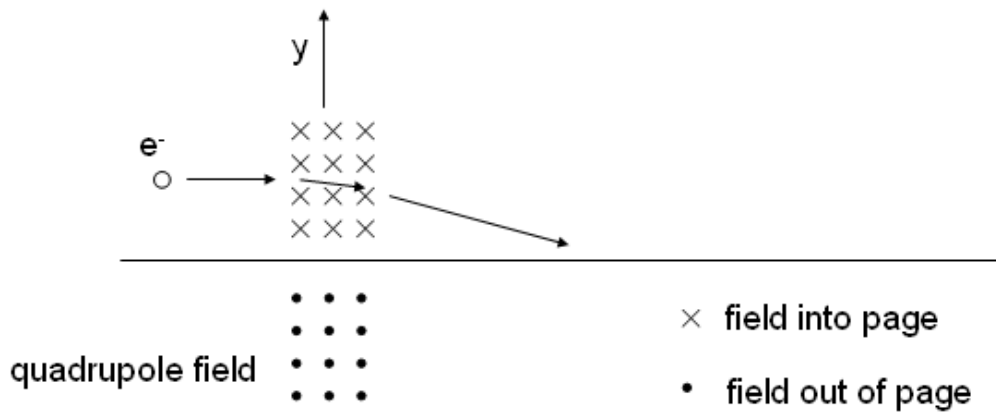


Figure 3: The magnetic force causes an electron to bend towards the path.

Using Fleming's left hand rule, you can work out that if an electron travelling into the page goes through the field somewhere on the positive y axis, it would be bent down. This is shown in fig. 3. If it goes through on the negative y axis, it would be bent up. In this way, a beam of electrons going through the field around the origin would be bent towards the z axis.

There is, however, a catch. If an electron goes through on the positive or negative x axis, it would be bent away from the z axis. That is, it would be defocused! This is shown in fig. 4.

So the quadrupole field is focusing in the y direction, but defocusing in the x. That seems to defeat the purpose. However, all is not lost. There is still a way to get it to work.

A quadrupole that is focusing in the x direction can be placed some

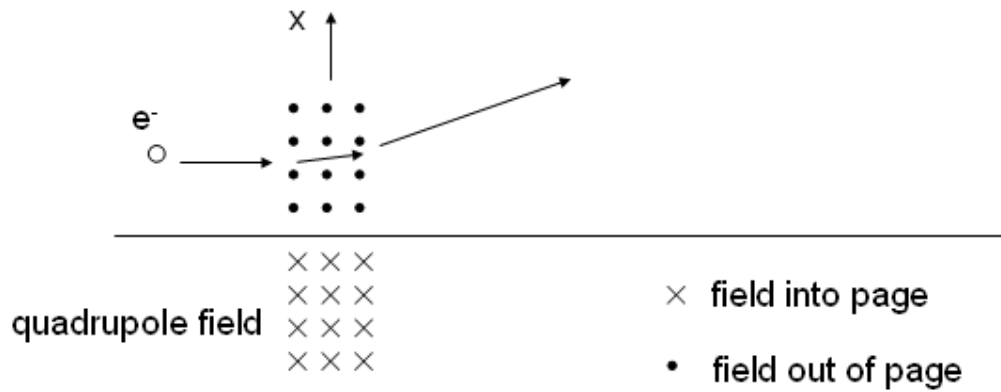


Figure 4: The magnetic force of the other axis causes an electron to bend away from the path.

distance away. Such a quadrupole must, of course be defocusing in the y direction. If the strengths of the two quadrupoles are equal, they would be focusing in both x and y direction. This is shown in fig. 5.

It is similar in optics. For lenses - one converging and one diverging of the same strength - placed at a distance of less than their focal length apart.

The transfer matrix for the quadrupole is given by:

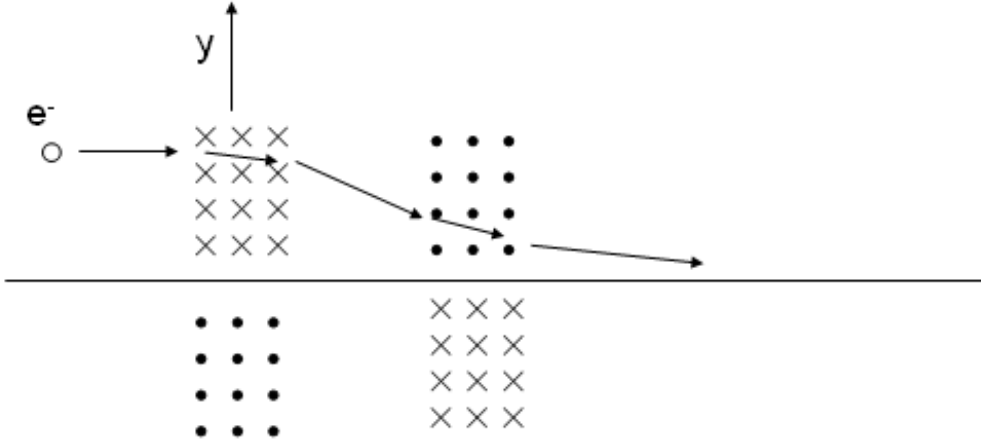


Figure 5: A focusing and defocusing quadrupole together can result in focusing.

$$\begin{pmatrix} x_1 \\ x'_1 \\ y_1 \\ y'_1 \\ z_1 \\ \delta_1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_{21} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{43} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \\ z_0 \\ \delta_0 \\ 1 \end{pmatrix} \quad (1)$$

where

$$m_{21} = -\frac{l}{B\rho} \frac{\partial B_y}{\partial x} \quad (2)$$

$$m_{43} = \frac{l}{B\rho} \frac{\partial B_y}{\partial x} \quad (3)$$

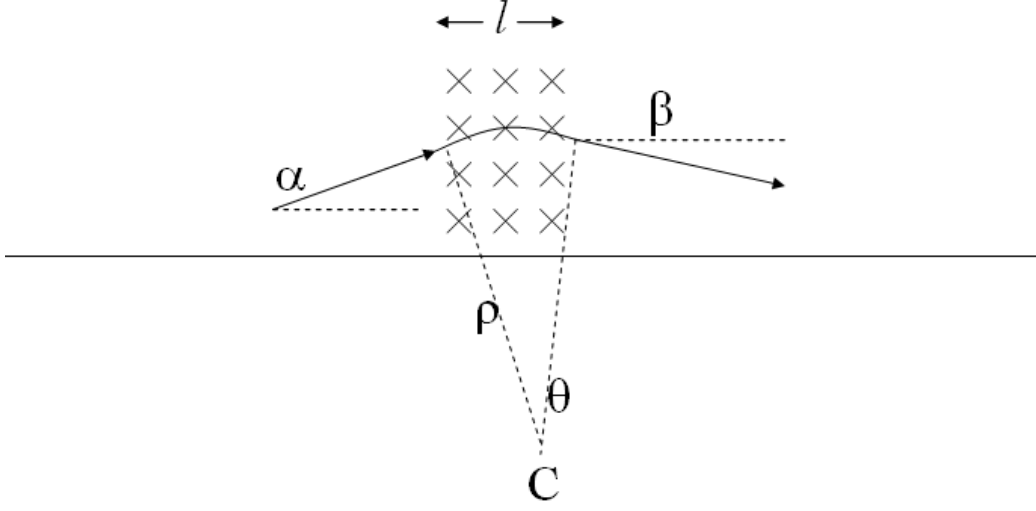


Figure 6: Bending of the electron path.

This gives the the following equations:

$$x'_1 = m_{21}x_0 + x'_0 \quad (4)$$

$$y'_1 = m_{43}y_0 + y'_0 \quad (5)$$

Substituting from the previous equations, the first equation is

$$x'_1 = -\frac{l}{B\rho} \frac{\partial B_y}{\partial x} x_0 + x'_0$$

We can understand this with the help of fig. 6.

$x'_0$  and  $x'_1$  are respectively  $\tan \alpha$  and  $\tan \beta$ . Typical beams are close to the axis, so that the gradients  $x'_0$  and  $x'_1$  are small. Therefore, these are close to  $\alpha$  and  $\beta$  respectively. The equation may then be written as

$$\beta = -\frac{l}{B\rho} \frac{\partial B_y}{\partial x} x_0 + \alpha.$$

The first term on the right hand side is then the change in angle as a result of the magnetic field. If the electron goes through the centre of the quadrupole,  $x_0$  is zero. So the change in angle is zero as we would expect, since the field is zero there.

The partial derivative  $\partial B_y/\partial x$  is the field gradient. Since the beam is close to the design path,  $x_0$  is small, and  $(\partial B_y/\partial x)x_0$  gives the magnetic field experienced by the electron. Let this be  $\delta B_y$ . The above equation is then

$$\beta = -\frac{l}{B\rho}\delta B_y + \alpha.$$

The expression  $B\rho$  is called the magnetic rigidity. If  $B$  is the magnetic field,  $\rho$  would be the radius of curvature of the electron path. It is a constant for a given momentum, because of the relation for magnetic force and centripetal force

$$\frac{\gamma m_e c^2}{\rho} = B e c.$$

If  $B$  is taken to be  $\delta B_y$ , then  $\rho$  would be the radius of the electron path in the field in fig. 6. Substituting  $\delta B_y$  for  $B$ , the above equation becomes

$$\beta = -\frac{l}{\rho} + \alpha.$$

In fig. 6, the fraction  $l/\rho$  is equal to the angle  $\theta$ , which is in turn equal to the difference between the angles  $\alpha$  and  $\beta$ . This means that

$$\beta = -\theta + \alpha,$$

which is the same as the previous equation. Working backwards, we can now recover one of the equations,

$$x'_1 = m_{21}x_0 + x'_0.$$

Applying the same reasoning to the defocusing field in the  $y$  axis, we would obtain the other equation,

$$y'_1 = m_{43}y_0 + y'_0.$$

## References

- [1] [http://en.wikipedia.org/wiki/Quadrupole\\_magnet](http://en.wikipedia.org/wiki/Quadrupole_magnet).
- [2] Andy Wolski's linear dynamics lecture 4, [http://hep.ph.liv.ac.uk/~hock/Good\\_References/Good\\_References.html](http://hep.ph.liv.ac.uk/~hock/Good_References/Good_References.html)